Relaxing the I.I.D. Assumption

Adaptively Minimax Optimal Regret via Root-Entropic Regularization

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Background
A Simple Example

Trading Stocks

• You need to invest your money into a stock portfolio.
• You have access to several market experts that give you advice.
• You regret not having always followed the post hoc best expert’s advice.

What assumptions should we make?

A simplifying assumption is that the data are I.I.D. (e.g., Black–Scholes–Merton).

In real life, market is driven in part by non-stochastic forces.

Is assuming adversarial data too pessimistic?

Is the departure from I.I.D.-ness benign? How can we quantify that?

Influence of non-stochastic forces “small” $\Rightarrow$ maybe.

Meaning of “small” TBD.

Want to maximize profit without having to know what drives the market.
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Sequential Prediction with Expert Advice

Sequential Prediction a.k.a. Online Learning

For rounds $t = 1, \ldots, T$:

- Receive $x(t) = (x_1(t), \ldots, x_n(t)) \subseteq \hat{Y}$ expert predictions
- Predict $\hat{y}(t) \in \hat{Y}$ based on historical data before time $t$
- Observe $y(t) \in Y$ from the environment
- Incur loss $\ell(\hat{y}(t), y(t))$
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- Given by the excess cumulative loss of the player over the best expert;

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\text{Regret: } R(T) = T \sum_{t=1}^{T} \ell(\hat{y}(t), y(t)) - \min_{i \in [N]} T \sum_{t=1}^{T} \ell(x_i(t), y(t))
\]

The prediction problem is online learnable if a player can incur sub-linear regret:

\[
\mathbb{E}R(T) \in o(T)
\]

Where the $\mathbb{E}$ is taken with respect to the randomness in the player’s and expert’s predictions, and the data-generating mechanism for $(y(t))_{t \in \mathbb{N}}$. (The $\mathbb{E}$ may be under a complicated, non-I.I.D. measure.)
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Optimality in the Stochastic and Adversarial Regimes

Stochastic-with-a-Gap

- Expert predictions and data are i.i.d. over time from some distribution.
- There is an expert whose mean loss is $\Delta$ smaller than the others.

Theorem (Gaillard et al. 2014 + Mourtada and Gaïffas 2019)

A constructive algorithm achieves the minimax regret:

$$E[R(T)] \approx \log N / \Delta,$$

uniformly bounded in $T$.

Adversarial

- Compete against expert predictions and data that maximize $R(T)$.

Theorem (Vovk 1998, see also [FS97; CL06])

A constructive algorithm achieves the minimax regret:

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for all $T$. Can a single algorithm be optimal in both settings simultaneously?
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Simultaneous Optimality of Hedge

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Proposition: [MG19] Hedge is optimal in both the stochastic and adversarial settings.

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Adversarial: $E[R(T)] \approx \sqrt{T \log N}$. 

Graph showing expected regret over time for adversarial and stochastic settings.
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Beyond Stochastic and Adversarial

Real data $\not\equiv$ stochastic.
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Real data $\not\equiv$ stochastic. $\leftarrow$ Too optimistic.

Building upon [RST11], we study a spectrum between stochastic and adversarial.

Intuitively, fix a “neighbourhood” of distributions. Each data point drawn from an arbitrary distribution in “neighbourhood”.
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**Definition BNR20**

An algorithm is **adaptively minimax optimal** for a spectrum of settings if:

- it achieves the minimax optimal performance in each setting; and
- it does not require knowledge of the true setting in advance.
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- If we know \( \theta \in \Lambda \) in advance, the best achievable performance is \( R^*_\theta(T) \).
- We want an algorithm that does as well as possible without knowing \( \theta \):

\[
R_\theta(T) \leq C R^*_\theta(T) \text{ uniformly in } \theta \text{ for large enough } T.
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Overview of Our Work

We show Hedge is suboptimal between Stochastic and Adversarial. This was surprising for us. We initially set out hoping to prove that Hedge was adaptive to all scenarios. Theorem BNR20 Without oracle knowledge to tune the learning rate, Hedge is not simultaneously minimax optimal at all settings between stochastic-with-a-gap and adversarial. We provide a new algorithm that achieves the minimax rate in all settings... ...without knowledge of which setting prevails!

Theorem BNR20 There is an adaptively minimax optimal algorithm: Meta-CARE.
Overview of Our Work

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We show Hedge is suboptimal between Stochastic and Adversarial. This was surprising for us. We initially set out hoping to prove that Hedge was adaptive to all scenarios.

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There is an adaptively minimax optimal algorithm: Meta-CARE.
Main Result

Motivating Intuition

• In the adversarial case the minimax optimal regret is $\Theta(\sqrt{T \log N})$

• If we know only $N_0$ of the experts can ever be "the best", and which ones,...

• we could restrict an adversarially optimal algorithm to the "best experts"

• so we might strive to have regret $\Theta(\sqrt{T \log N_0})$ in $(T, N_0)$

• If we know one expert is better than the rest by $\Delta_0$, but not which it is...

• then we are almost in the stochastic-with-a-gap case

• so we might hope for regret $\Theta((\log N) / \Delta_0)$

Theorem BNR20

The adaptively minimax optimal rate of regret, which Meta-CARE achieves, is $E R(T) \approx \begin{cases} \sqrt{T \log N_0} & N_0 \geq 2 \\ \frac{(\log N)}{\Delta_0} & N_0 = 1 \end{cases}$. 
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Relaxing the I.I.D. Assumption
Our Setting: Time-Homogeneous Convex Constraints

Intuition
Experts and environment may collude.

Realizations
\((x(t), y(t))\) are sampled from an adversarial conditional distribution.

Formal Framework
• Fix a convex set of distributions \(D \subseteq M(\hat{Y}^N \times Y)\).
• \((x(t), y(t))\) drawn from an element of \(D\) given the history prior to \(t\).
• The choice of distribution is made based on outcomes of the previous rounds.

More Details
• Time-Homogeneous: \(D\) does not depend on \(t\).
• Convex: environment can flip a coin to select between basic elements of \(D\).
• Environment may aim to maximize regret subject to the constraint.
Our Setting: Time-Homogeneous Convex Constraints

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The set $\mathcal{D}$ may be complex and difficult to describe for a particular application.

Effective Experts

$I_0 = \{\text{experts that are optimal in } \mathcal{E} \text{ for some } \mu \in \mathcal{D}\}$

$N_0 = |I_0|$ Analogous to the single best expert in the stochastic-with-a-gap setting.

Effective Stochastic Gap

$\Delta_0 = \inf_{\mu \in \mathcal{D}} \{\mu - \text{expected difference in loss of best expert and best expert not in } I_0\}$

Analogous to the gap in the stochastic-with-a-gap setting.
The set $\mathcal{D}$ may be complex and difficult to describe for a particular application.

We want to characterize the constraint using quantities that:

- simplify the abstract complexity of the constraint;
- differentiate whether the data is “easy” or not, independent of algorithms;
- yield matching lower and upper bounds on regret.

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**Setting:** the means for each expert are jointly defined by a parameter \( \alpha \), 
\[ N = 5, \mathcal{I}_0 = \{1, 3, 5\}, N_0 = 3. \]
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Examples

Stochastic-with-a-gap:

\[ D = \{ \mu_0 \} \]

- \( N_0 = 1 \),
- \( I_0 = \{ i^* = \arg\min_{i \in [N]} E \mu_0[\ell_i] \} \),
- \( \Delta_0 = \min_{i \neq i^*} E \mu_0[\ell_i - \ell_{i^*}] \)

Adversarial:

\[ D = M(\hat{Y} \times Y) \]

- \( N_0 = N \),
- \( \Delta_0 = +\infty \)

Adversarial-with-an-E-gap (Mourtada and Gaïffas 2019)

- All measures where a common expert is better than others in \( E \) by \( \Delta > 0 \).
- By design, \( N_0 = 1 \) and \( \Delta_0 = \Delta \).

Neighborhood-of-I.I.D.

- Pick any distribution \( \mu_0 \), and any radius, \( r \geq 0 \).

\[ D = \text{Ball}(\mu_0, r) \]

- Suppose that \( \mu_0 \) has a gap between each of the mean losses.
- \( N_0, \Delta_0 \) depend on the radius of the ball...
Examples

Stochastic-with-a-gap: $\mathcal{D} = \{\mu_0\}$,
Examples

Stochastic-with-a-gap: \( D = \{ \mu_0 \} \),

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Examples

**Stochastic-with-a-gap:** \( \mathcal{D} = \{ \mu_0 \} \),

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**Adversarial:** $\mathcal{D} = \mathcal{M}(\hat{\mathcal{Y}}^N \times \mathcal{Y})$
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Adversarial: \( \mathcal{D} = \mathcal{M}(\hat{Y}^N \times \mathcal{Y}) \leftarrow \) contains point masses!
Examples

Stochastic-with-a-gap: $\mathcal{D} = \{\mu_0\}$,
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Adversarial: $\mathcal{D} = \mathcal{M}(\hat{\mathcal{Y}}^N \times \mathcal{Y}) \leftarrow$ contains point masses!
- $\mathcal{N}_0 = \mathcal{N}$. 
Examples

Stochastic-with-a-gap: \( \mathcal{D} = \{\mu_0\} \),
- \( N_0 = 1 \), \( \mathcal{I}_0 = \left\{ i^* = \arg\min_{i \in [N]} \mathbb{E}_{\mu_0}[\ell_i] \right\} \), \( \Delta_0 = \min_{i \neq i^*} \mathbb{E}_{\mu_0}[\ell_i - \ell_{i^*}] \)

Adversarial: \( \mathcal{D} = \mathcal{M}(\hat{\mathcal{Y}}^N \times \mathcal{Y}) \) ← contains point masses!
- \( N_0 = \mathcal{N} \), \( \Delta_0 = +\infty \)
Examples

Stochastic-with-a-gap: \( \mathcal{D} = \{ \mu_0 \} \),

- \( N_0 = 1, \ I_0 = \left\{ i^* = \arg \min_{i \in \mathcal{N}} \mathbb{E}_{\mu_0}[\ell_i] \right\}, \ \Delta_0 = \min_{i \neq i^*} \mathbb{E}_{\mu_0}[\ell_i - \ell_{i^*}] \)

Adversarial: \( \mathcal{D} = \mathcal{M}(\hat{Y}^N \times \mathcal{Y}) \leftarrow \text{contains point masses!} \)

- \( N_0 = \mathcal{N}, \ \Delta_0 = +\infty \)

Adversarial-with-an-\( \mathbb{E} \)-gap (Mourtada and Gaïffas 2019)
Examples

Stochastic-with-a-gap: \( \mathcal{D} = \{\mu_0\} \),
- \( N_0 = 1 \), \( I_0 = \left\{ i^* = \arg\min_{i \in [N]} E_{\mu_0}[\ell_i] \right\} \), \( \Delta_0 = \min_{i \neq i^*} E_{\mu_0}[\ell_i - \ell_{i^*}] \)

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- All measures where a common expert is better than others in \( E \) by \( \Delta > 0 \).
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**Adversarial-with-an-$\mathbb{E}$-gap** (Mourtada and Gaïffas 2019)
- All measures where a common expert is better than others in $\mathbb{E}$ by $\Delta > 0$.
- By design, $N_0 = 1$ and $\Delta_0 = \Delta$. 

Neighborhood-of-I.I.D.
- Pick any distribution $\mu_0$, and any radius, $r \geq 0$. $\mathcal{D} = \text{Ball}(\mu_0, r)$
- Suppose that $\mu_0$ has a gap between each of the mean losses.
- $N_0$, $\Delta_0$ depend on the radius of the ball...
Stochastic-with-a-gap: \( \mathcal{D} = \{\mu_0\} \),

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Interpreting \((\mathcal{N}_0, \Delta_0^{-1})\)

\[ \mathcal{D} = \text{Ball}(\mu, \text{radius}) \text{ w/ } \mathbb{E}_\mu \ell_1 < \mathbb{E}_\mu \ell_2 < \ldots \]
Interpreting $(\mathcal{N}_0, \Delta_0^{-1})$

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Interpreting \((N_0, \Delta_0^{-1})\)

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- \(\Delta_0\) increases between \(N_0\) jumps
- \(\Delta_0\) resets at each jump

Lexicographical order on \((N_0, \Delta_0^{-1})\)

- For nested \(D\)s, larger one is "harder" to learn.

\((N_0, \Delta_0^{-1})\) quantifies the difficulty.
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How does regret change with $\Delta_0^{-1}$?
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Lexicographical order on \( \mathcal{(N}_0, \Delta_0^{-1}) \)
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- For nested \(D\)s, larger one is “harder” to learn.
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- How does regret change with \(\Delta_0\)?
Impact of \((N_0, \Delta_0^{-1})\) on Regret

\begin{center}
\begin{figure}
\centering
\includegraphics[width=\textwidth]{graph.png}
\end{figure}
\end{center}

- \(N = N = 1000\) (Hedge)
- \(N_0 = 1\) (Hedge)
Impact of \((N_0, \Delta_0^{-1})\) on Regret

\[
T_0 \approx \frac{\log N}{(\Delta_0)^2}
\]

\[
\sqrt{T_0 \log N} \approx \frac{(\log N)}{(\Delta_0)}
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Impact of \( (N_0, \Delta_0^{-1}) \) on Regret

\[
\begin{align*}
T_0 & \approx \frac{\log N}{\Delta_0^2} \\
T & \approx \left( \log N \right) / (\Delta_0)
\end{align*}
\]
Impact of \((N_0, \Delta_0^{-1})\) on Regret

![Graph showing the impact of \((N_0, \Delta_0^{-1})\) on regret over time. The graph plots expected regret against time for different values of \(N_0\) and \(N\). The y-axis represents expected regret, and the x-axis represents time. The legend indicates the different cases: \(N_0 = N = 1000\) (Hedge), \(N_0 = 2\) (Hedge), \(N_0 = 1\) (Hedge), \(N_0 = 2\) (Meta-CARE), \(N_0 = 1\) (Meta-CARE).]
Performance of Hedge
Hedge Algorithm

We will consider only finite expert classes and bounded losses \( \ell : \hat{\mathcal{Y}} \times \mathcal{Y} \rightarrow [0, 1] \).
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Hedge Algorithm

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A proper algorithm assigns probability $w_i(t)$ to expert $i$ at time $t$. 
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**Hedge Algorithm**

• Fix learning rate schedule $\eta : \mathbb{N} \rightarrow \mathbb{R}$; initialize the weights as uniform; define
  $$\ell_i(t) = \ell(x_i(t), y(t)),$$
  $$L_i(t) = \sum_{s=1}^{t} \ell_i(s).$$

• Update weights for each $i \in [N]$ using
  $$w_i(t) \propto \exp\{-\eta(t)L_i(t-1)\}.$$
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Hedge Regret Bounds

Consider playing Hedge with $\eta(t) = c/\sqrt{t}$ for any convex $\mathcal{D}$.

Recall:
- $N_0$ is the number of effective experts,
- $\Delta_0$ is the effective stochastic gap.
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**Theorem BNR20**

Taking $c \propto \sqrt{\log N}$ gives

$$
\mathbb{E}R(T) \lesssim \begin{cases} 
\sqrt{T \log N} + \frac{\log N}{\Delta_0} & : N_0 \geq 2 \\
(\log N)/\Delta_0 & : N_0 = 1.
\end{cases}
$$

Taking $c \propto 1$ gives

$$
\mathbb{E}R(T) \lesssim (\log N_0)\sqrt{T} + \frac{(\log N)^2}{\Delta_0}
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We also prove matching lower bounds!
Hedge Regret Bounds

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**Theorem BNR20**

If the player has oracle knowledge of $N_0 > 1$, taking $c \propto \sqrt{\log N_0}$ gives

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In all three cases, we interpret terms involving...
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In all three cases, we interpret terms involving...
- $T$: long run regret accumulation after adapting
- $(\log N, \Delta_0)$: adversarial regret over adaptation period of duration $O\left(\frac{(\log N)^2}{c^2 \Delta_0^2}\right)$
Can we do better than Hedge?

**Question:** If we don’t know $N_0$, can we learn adaptively and minimax optimally?

**Answer:**

Yes! We introduce two new algorithms in order to do this:

- **FTRL-CARE**, accomplished 1 and 3, but not 2.
- Slightly worse dependence on $N$.
- **Meta-CARE**, accomplished all 3 by boosting FTRL-CARE with Hedge.
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In particular, is there an algorithm for which...

1. $(T, N_0)$-dependence matches Hedge with oracle knowledge of $N_0$,
2. $(\log N, \Delta_0)$-dependence optimal for the stochastic case when $N_0 = 1$, and
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- FTRL-CARE, accomplished 1 and 3, but not 2.
  - slightly worse dependence on $N$.
- Meta-CARE, accomplished all 3 by boosting FTRL-CARE with Hedge.
Improved Algorithms and Bounds
Intuition for Improving on Hedge

Three Key Insights:

1. From our oracle Hedge bound, we want a learning rate \( \propto \sqrt{\log N_0 / t} \).

2. The regret of Hedge closely depends on the entropy of the weights:
   \[ H(w) = -\sum_{i \in [N]} w_i \log(w_i) \].

3. Worst-case adversary forces weights to concentrate to \( \text{Unif}(I_0) \), so
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What if we could have our learning rate at time \( t \), \( \eta(t) \), look like
\[
\eta(t) = \sqrt{\frac{H(w(t))}{t}}
\]
Follow the Regularized Leader

FTRL is a fundamental online linear optimization algorithm.
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\[
w(t + 1) \in \arg\min_{w \in \text{simp}([N])} \left( \langle w, L(t) \rangle - \frac{\sqrt{t+1}}{c_1} \sqrt{H(w) + c_2} \right).
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CARE if you can, Hedge if you must; or, Meta-CARE for All
FTRL-CARE is *almost* adaptively minimax optimal.

**Theorem BNR20**

For any convex $\mathcal{D}$, FTRL-CARE achieves

$$\mathbb{E} R(T) \lesssim \sqrt{T \log N_0} + \frac{(\log N)^{3/2}}{\Delta_0}.$$  

When $N_0 = 1$, it incurs total regret of order $\frac{(\log N)^{3/2}}{\Delta_0}$ instead of $\frac{(\log N)}{\Delta_0}$. 
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Proof Details

Proof Technique 1:

FTRL-CARE looks like Hedge with oracle knowledge.

Lemma BNR20

FTRL-CARE is equivalent to solving the following system of equations:

\[ \eta (t + 1) = c_1 \sqrt{H (w (t + 1))} + c_2 t + 1 \]

and

\[ w (t + 1) = \exp \left\{ -\eta (t + 1) L (t) \right\} \sum_{i \in [N]} \exp \left\{ -\eta (t + 1) L_i (t) \right\} \].

Proof Technique 2:

Concentration of measure holds under our relaxation of i.i.d.

Lemma BNR20

For any prediction algorithm, constraint \( D \), and data-generating mechanism, \( \sup_{i \in [N]} I_{0} E_{\min_{i \in I_{0}}} \exp \left\{ \lambda T \sum_{t=0}^{T} [\ell_{i_0} (t) - \ell_i (t)] \right\} \leq \exp \left\{ T \left( \frac{\lambda^2}{2} - \lambda \Delta_0 \right) \right\} \).
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Summary
Our Contributions

1. Introduced a spectrum of relaxations of the I.I.D. assumption.
   • Data that we want to predict won’t be purely adversarial or stochastic.
   • We want to know that we do well in intermediate scenarios as well.

2. Characterized minimax regret under time-homogeneous convex constraints.
   • Depends on the number of effective experts, $N_0$, and the effective stochastic gap, $\Delta_0$.

3. Formalized the notion of adaptive minimax optimality.

4. Proved Hedge is not adaptively minimax optimal along spectrum from I.I.D. to adversarial.
   • Requires oracle knowledge to get minimax optimal dependence on $T$ and $N_0$.

5. Provided a new algorithm, Meta-CARE that is adaptively minimax optimal.
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