Relaxing the I.I.D. Assumption

Adaptively Minimax Optimal Regret via Root-Entropic Regularization

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[*] Equal Contribution

[†] Presenting

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Background

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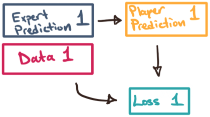

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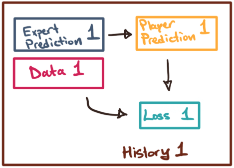

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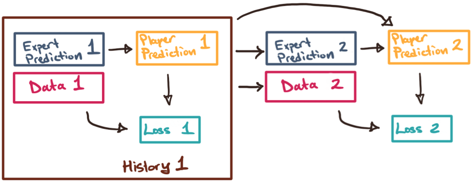

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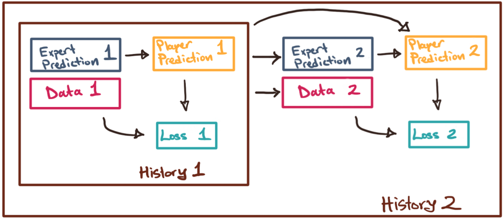

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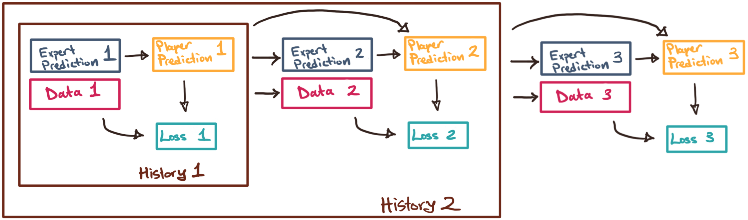

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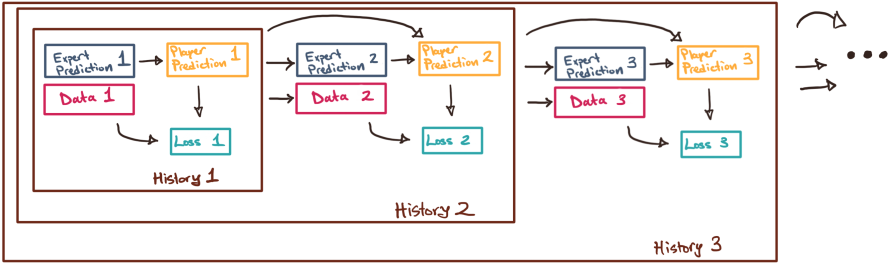

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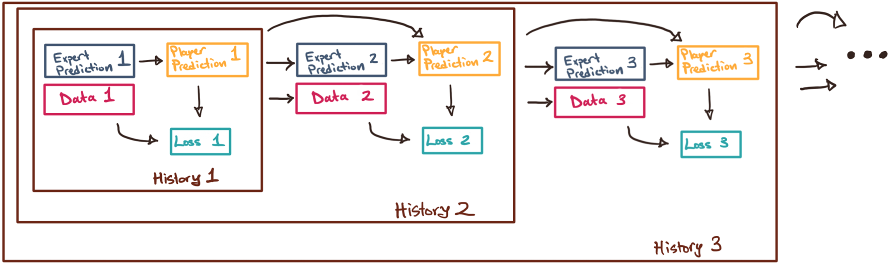

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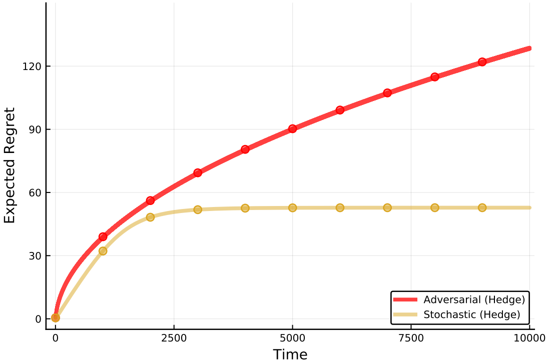

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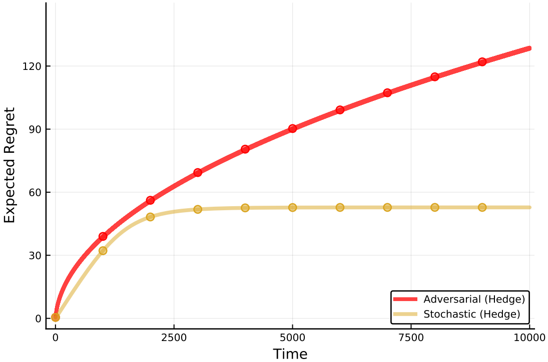

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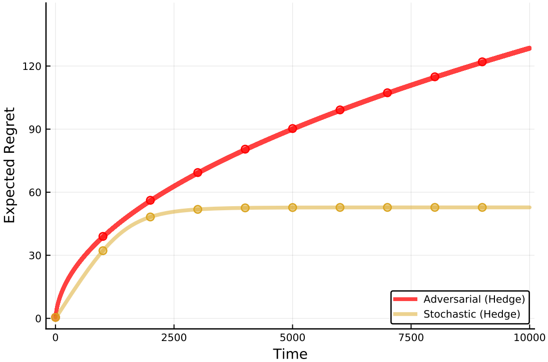

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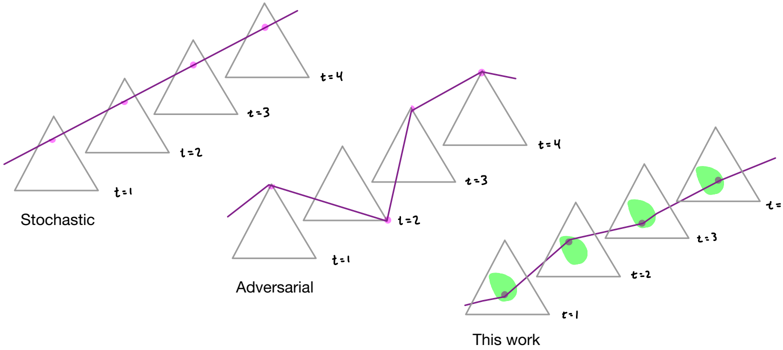

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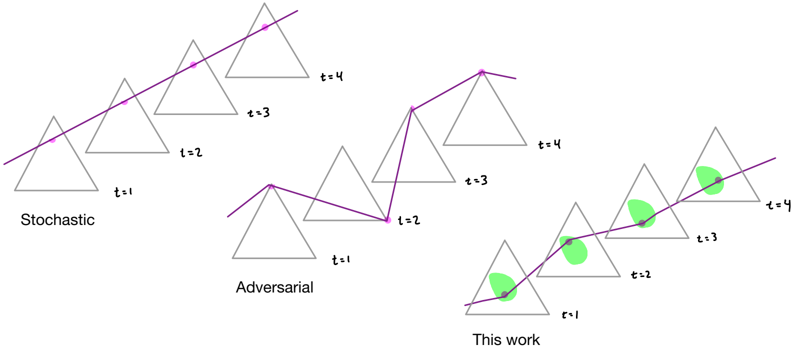

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Relaxing the I.I.D. Assumption

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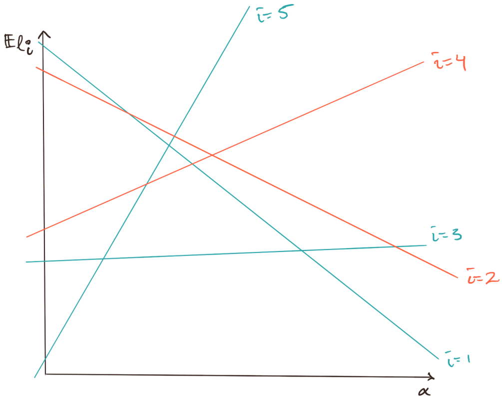

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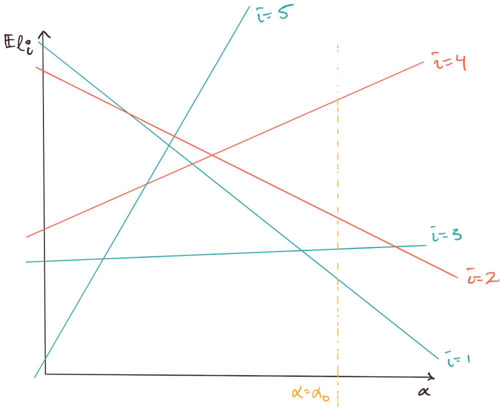

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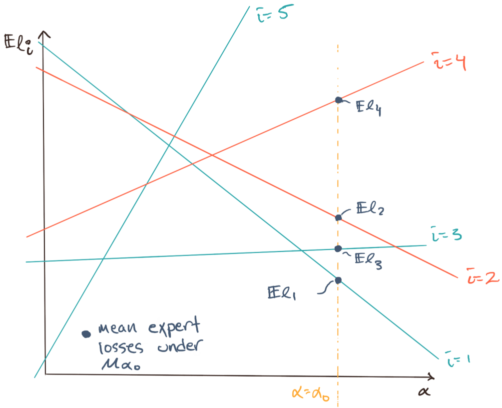

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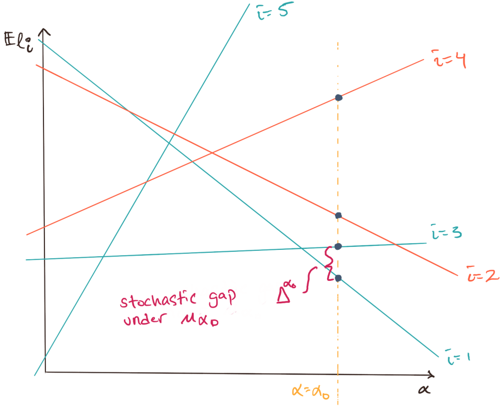

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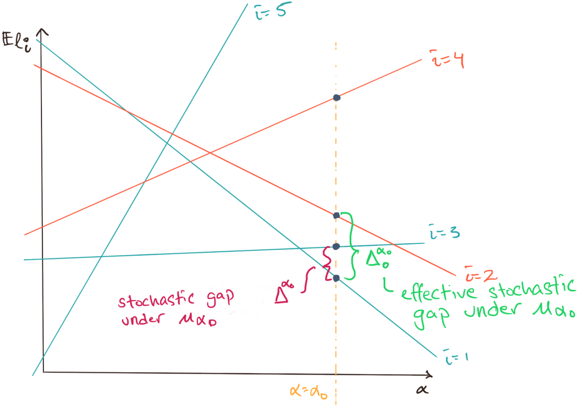

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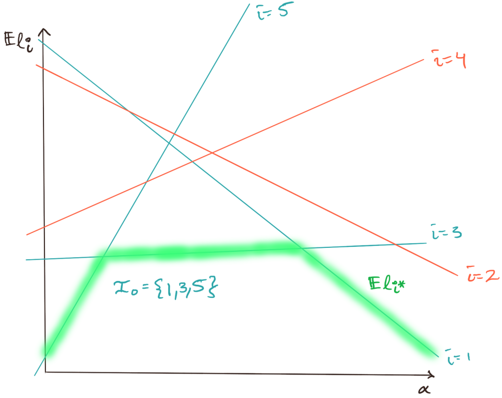

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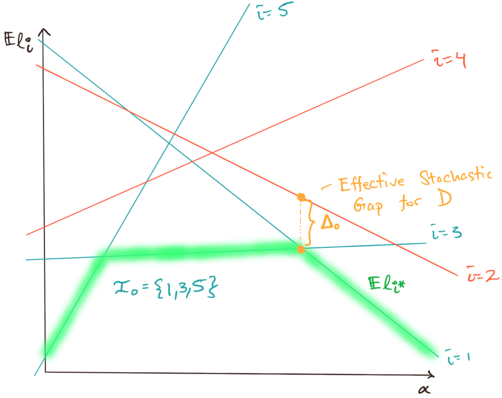

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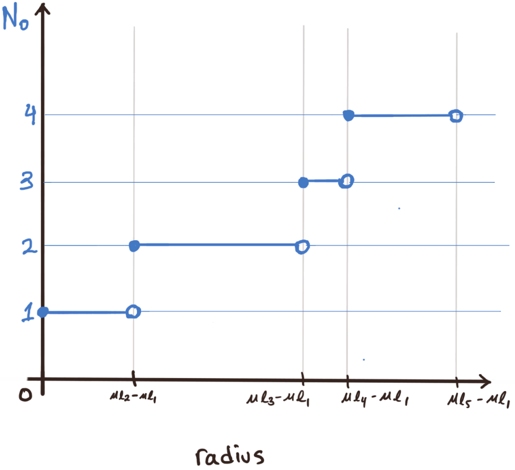

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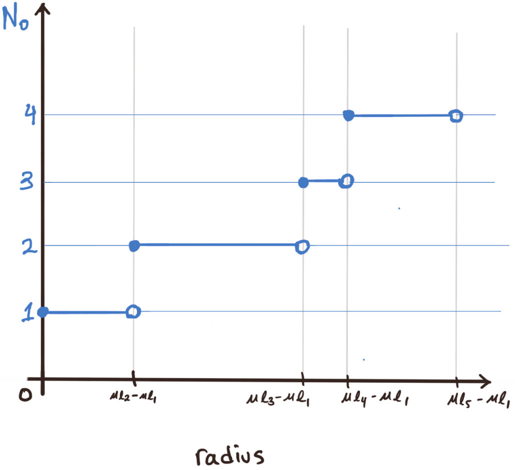

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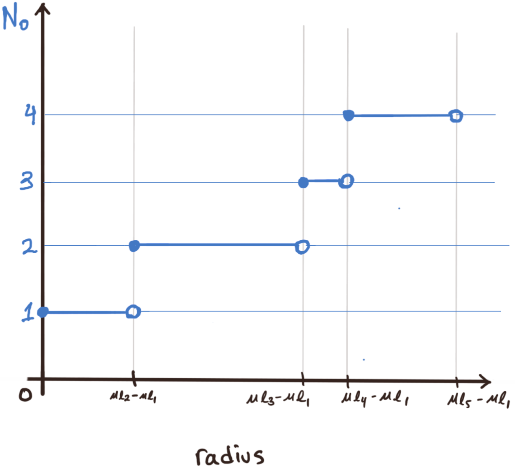

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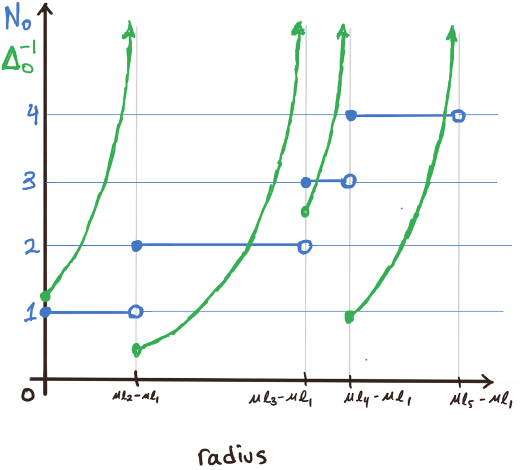

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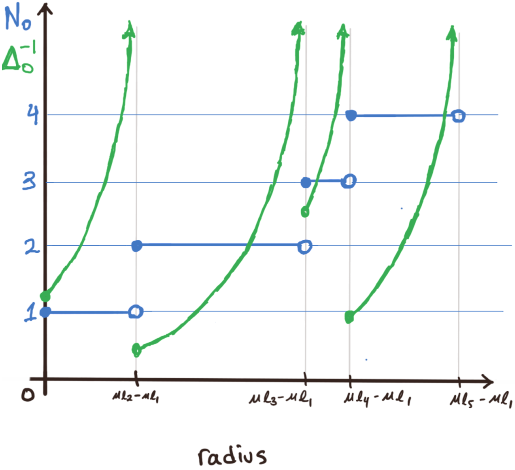

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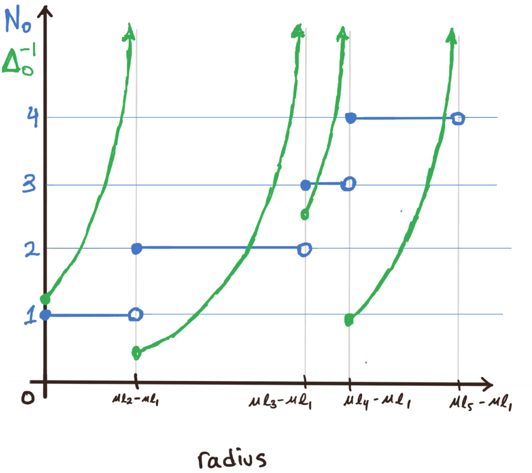

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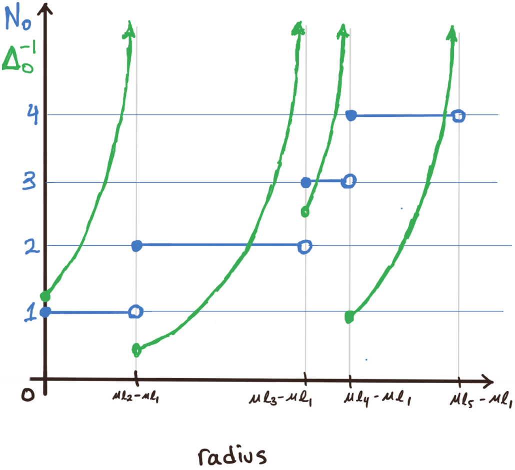

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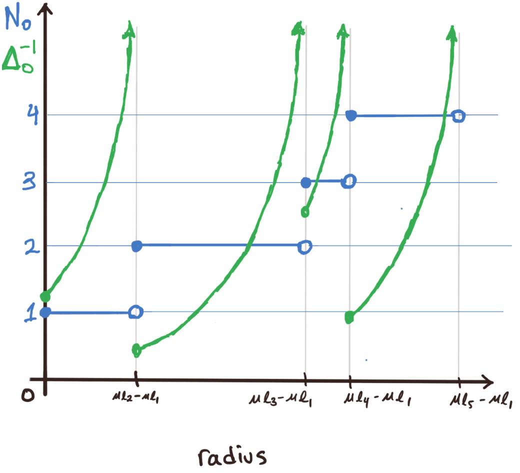

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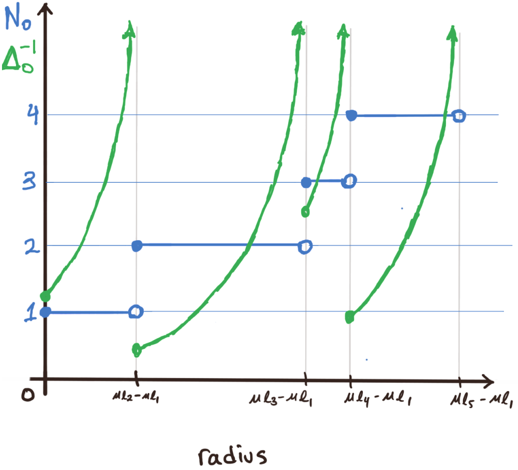

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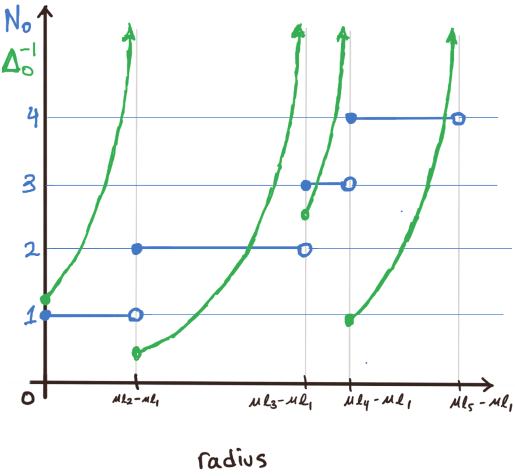

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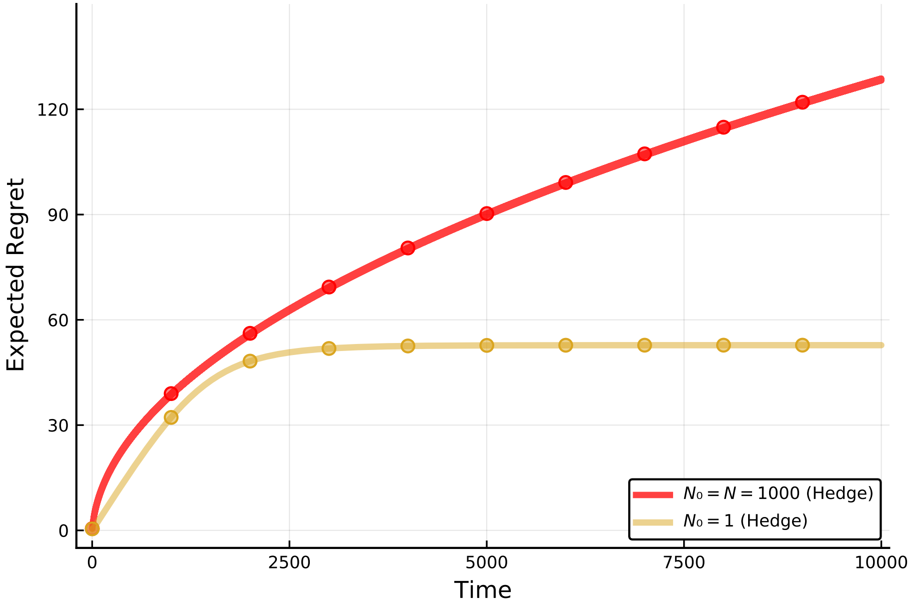
Impact of (N_0, Δ_0^{-1}) on Regret



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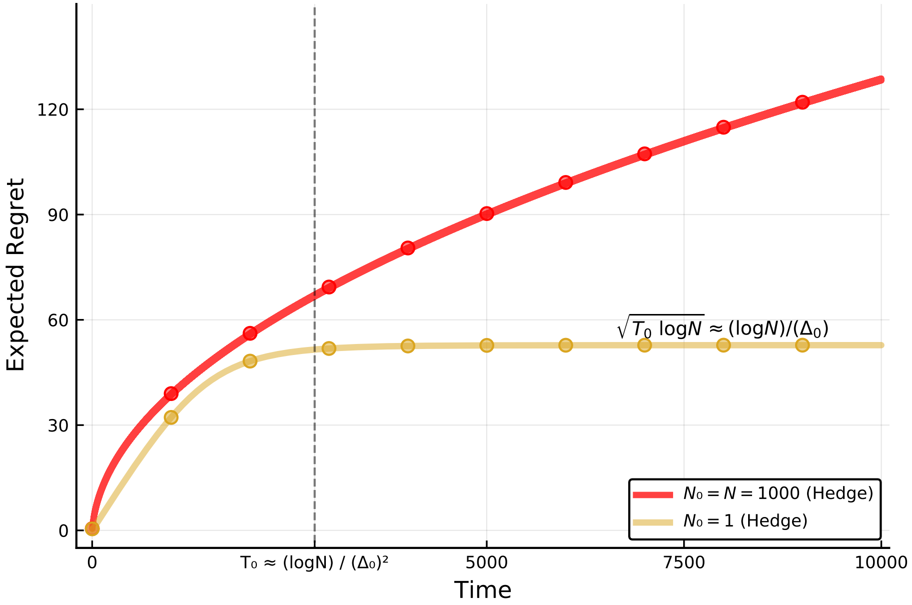

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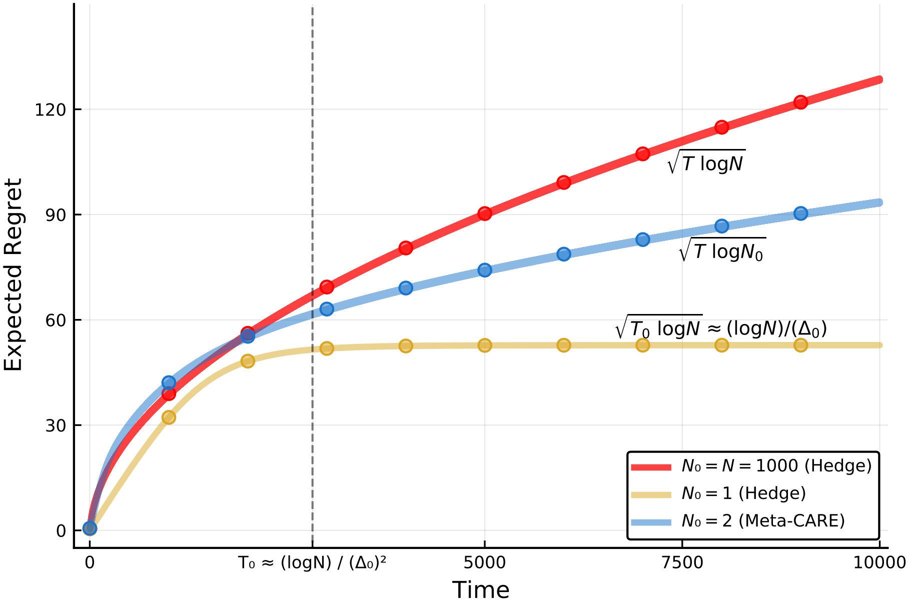

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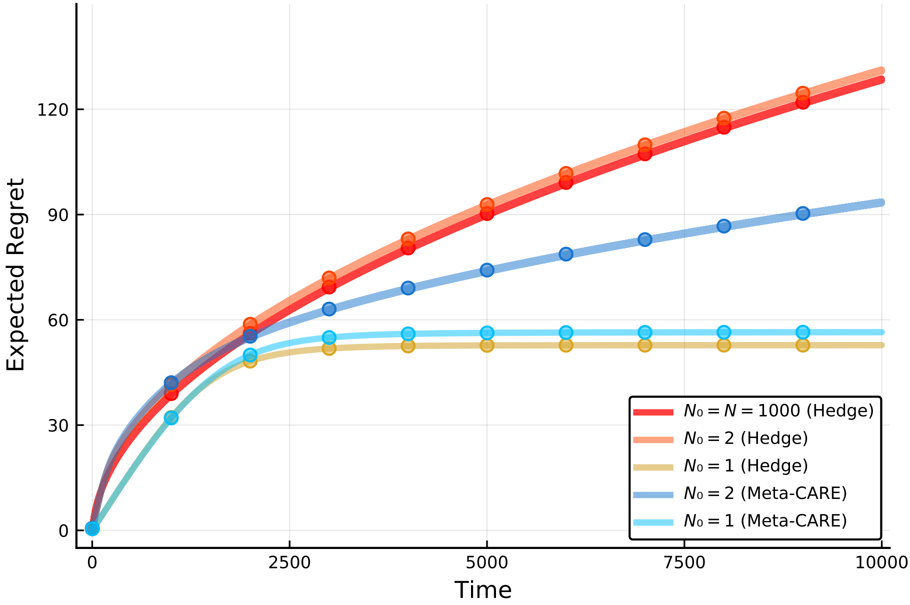

Performance of Hedge

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What if we could have our learning rate at time t, $\eta(t)$, look like

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